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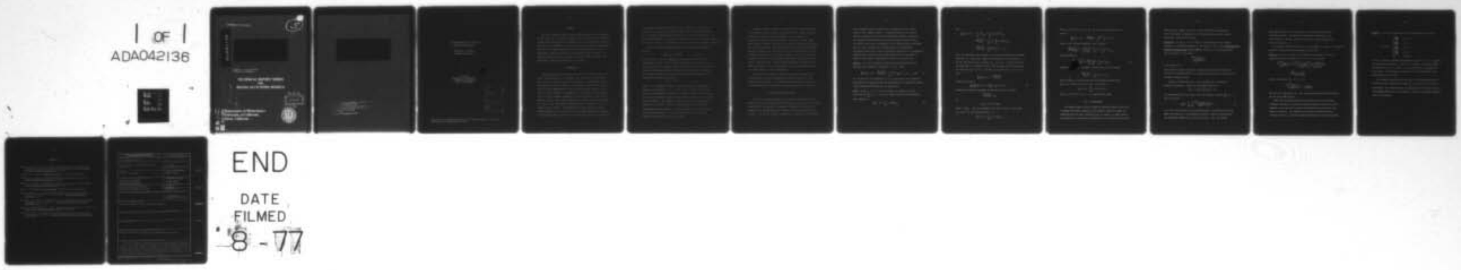

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A Characterization of Convoluted Geometric Distributions

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SUMMARY

Let X be a nonnegative integer-valued random variable whose distribution is that of the sum of a geometric variable Y with parameter θ and a nonnegative integer-valued random variable Z independent of Y whose distribution does not depend on θ. The variable X is said to have a convoluted geometric distribution. The class of all such distributions are here characterized by a system of differential equations satisfied by their probability mass functions. The characterization is shown to be useful in maximum likelihood estimation of the parameter θ when the distribution of Z is known.

I. INTRODUCTION

There are many sources in nature that give rise to discrete data in the form of signals plus noise. For example, data obtained by a Geiger counter may be viewed as sums of counts due to the presence of a radioactive substance and counts due to noise or static. Other examples in which discrete signal plus noise models appropriately describe random phenomena occur in the use of sonar for bottom fishing (see Cushing (1973), for example), in physiological processes such as synaptic transmission of neural impulses (see Katz (1966), Samaniego (1976)), and in single server queues in which individuals arriving for service can be classified into mutually exclusive categories (see, for example, Shonick (1970)). In situations such as these when the observed random variable presents itself as the sum of independent discrete variables, its distribution may be described as a discrete signal plus noise distribution.

A one parameter distribution might be viewed as appropriate when the noise variable can be observed alone, but the signal, whose distribution belongs to a one parameter family, can be observed only in the presence of noise. The noise distribution may be estimated with high precision based on auxiliary sampling on the noise variable, and thus might be assumed equal to the empirical distribution.

Let Y be a geometric random variable with probability mass function given by

$$P_\theta(Y = k) = (1-\theta)\theta^k, \qquad k = 0,1,2,\ldots$$

where $\theta \in (0,1)$. If Z is a nonnegative integer-valued random variable independent of Y, and $X = Y + Z$, then the probability mass function of X is the convolution of the sequences $\{P_\theta(Y = i)\}$ and $\{P(Z = i)\}$. Such convoluted geometric distributions arise in a familiar life testing context. Suppose the lifetime (in hours) of an item on test is distributed according to an exponential distribution with density

$$f_\lambda(x) = \lambda e^{-\lambda x} \qquad x > 0$$

where $\lambda > 0$. The number of hours of life of the item has a geometric distribution with parameter $\theta = e^{-\lambda}$. If the items on test are not inspected hourly, and the lifetime is recorded as the number of hours before the wear-out is observed, then the recorded lifetime might be modelled as a convoluted geometric variable. For example, if inspection occurs independently with probability p each hour. Then the recorded lifetime is the sum of two independent geometric variables. If the recording of a wear-out is delayed one hour with probability p, then the recorded lifetime is the sum of independent geometric and Bernoulli variables.

Estimation problems for specific signal plus noise distributions have been examined by several authors. For example, Gaffey (1959) constructed a consistent extimator for the distribution of one component of a continuous signal plus noise distribution. Sclove and Van Ryzin (1969) derived method of moments estimators for a variety of multiparameter signal plus noise distributions. There has been only limited success in maximum likelihood estimation for such models due to the cumbersome nature of the likelihood function, which, for discrete signals, consists of the product of (possibly infinite) sums involving the probability mass function of the signal. Samaniego (1976) examined maximum likelihood estimation for one parameter convoluted Poisson distributions, solving the estimation problem for certain families of such distributions.

In this paper, the family of convoluted geometric distributions is characterized by a system of differential equations satisfied by their probability mass functions. An application of the characterization result to maximum likelihood estimation of the geometric parameter is presented.

II. THE CHARACTERIZATION RESULT

A variety of characterizations of the geometric distribution, or more generally, the negative binomial distribution, may be found in the literature. Ferguson (1965) characterizes the geometric distribution in terms of the independence of the minimum and the difference of two independent discrete variables. The negative binomial distribution is characterized by Katz(1946) via a difference equation satisfied by its probability mass function together with a moment condition. A characterization of the negative binomial distribution by systems of differential equations satisfied by its probability mass function was obtained by Boswell and Patil (1973). This latter result bears a resemblance to the characterization established here, but does not reduce to a characterization of the geometric distribution alone and is based on differential equations of a substantially different form. Moreover, their result hypothesizes the existence of moments of all orders, a requirement not made in the result below.

Theorem. Let X be a nonnegative integer-valued random variable whose distribution is indexed by a parameter $\theta \in (0,1)$. Then

$$\frac{\partial}{\partial\theta} P_\theta(X = n) = -\frac{P_\theta(X = n)}{1-\theta} + \sum_{j=1}^{n} \theta^{j-1} P_\theta(X = n-j) \qquad \forall n, \forall\theta \tag{1}$$

if, and only if, the distribution of X is a convolution of the geometric distribution with parameter θ and the distribution of a nonnegative integer valued random variable which does not depend on θ.

Proof. Let $\{p_i\ , i = 0,1,\ldots\}$ be a sequence of constants such that $p_i \in [0,1] \forall i$ and $\sum_{i=0}^{\infty} p_i = 1$. Let X be a random variable whose probability mass function is

$$P_\theta(X = n) = \sum_{i=0}^{n} (1-\theta)\theta^i p_{n-i}\ . \tag{2}$$

Then

$$\frac{\partial}{\partial\theta} P_\theta(X = n) = -\sum_{i=0}^{n} \theta^i p_{n-i} + \sum_{i=1}^{n} i(1-\theta)\theta^{i-1} p_{n-i}$$

$$= -\frac{P_\theta(X = n)}{1-\theta} + \sum_{j=1}^{n} \theta^{j-1} \sum_{i=j}^{n} (1-\theta)\theta^{i-j} p_{n-i}$$

$$= -\frac{P_\theta(X = n)}{1-\theta} + \sum_{j=1}^{n} \theta^{j-1} P_\theta(X = n-j).$$

Thus, the probability mass function of a convoluted geometric distribution satisfies (1). Conversely, suppose the distribution of X satisfies (1). We show that there exists a sequence $\{p_i , i = 0,1,\ldots\}$, with $p_i \in [0,1]$ $\forall i$, such that $P_\theta(X = n)$ satisfies (2) $\forall n$. The fact that $\Sigma\, p_i = 1$ follows from the equation $\Sigma\, P_\theta(X = n) = 1$. The proof is by induction. For n = 0, we have

$$\frac{\partial}{\partial\theta} P_\theta(X = 0) = -\frac{P_\theta(X = 0)}{1-\theta}$$

which may be written

$$\frac{1}{1-\theta}\frac{\partial}{\partial\theta} P_\theta(X = 0) + \frac{1}{(1-\theta)^2} P_\theta(X = 0) = 0 \tag{3}$$

Equating the indefinite integrals of both sides of (3), we obtain

$$\frac{P_\theta(X = 0)}{1-\theta} = p_0$$

or

$$P_\theta(X = 0) = (1-\theta)p_0.$$

Since $(1-\theta)p_0$ must be a probability for all $\theta \in (0,1)$, we have that $p_0 \in [0,1]$. We now assume that for $r < k$,

$$P_\theta(X = r) = \sum_{i=0}^{r} (1-\theta)\, \theta^i p_{r-i}$$

where $p_j \in [0,1]$ for $j = 0,1,\ldots,r$, and show that (2) holds for $n = k$. We have

$$\frac{\partial}{\partial\theta} P_\theta(X = k) = -\frac{P(X = k)}{1 - \theta} + \sum_{j=1}^{k} \theta^{j-1} P_\theta(X = k - j)$$

which, by the induction hypothesis, may be written

$$\frac{\frac{\partial}{\partial\theta} P_\theta(X = k)}{1 - \theta} + \frac{P_\theta(X = k)}{(1 - \theta)^2} = \sum_{j=1}^{k} \theta^{j-1} \sum_{i=j}^{k} \theta^{i-j} p_{k-i},$$

which simplifies to

$$\frac{\frac{\partial}{\partial\theta} P_\theta(X = k)}{1 - \theta} + \frac{P_\theta(X = k)}{(1 - \theta)^2} = \sum_{i=1}^{k} i\, \theta^{i-1} p_{k-i}. \tag{4}$$

Equating the indefinite integrals of both sides of (4), we obtain

$$\frac{P_\theta(X = k)}{1 - \theta} = \sum_{i=1}^{k} \theta^{i} p_{k-i} + p_k.$$

The fact that $p_k \in [0,1]$ follows from the boundary condition that $P_\theta(X = k) \in [0,1]$ for all $\theta \in (0,1)$. We thus have

$$P_\theta(X = k) = \sum_{i=0}^{k} (1 - \theta)\, \theta^{i} p_{k-i}$$

with $p_j \in [0,1]$ for $j = 0,1,\ldots,k$, completing the proof.

III. AN APPLICATION

The natural domain in which to apply the characterization result above is maximum likelihood estimation of the geometric parameter θ under the assumption that the noise distribution $\{p_i\}$ is known. As stated earlier, the assumption of a known noise distribution may be realized when the noise variable may be sampled by itself. Maximum likelihood estimation of θ may be facilitated by application of the characterization result when the monotonicity property below obtains.

Definition. Let $\{P_\theta, \theta \in \Theta\}$ be a family of distributions on the integers indexed by a real-valued parameter θ. The family is said to have strong parametric monotone decreasing ratio (strong PMDR) if for any $a < b$ receiving positive mass for every $\theta \in \Theta$, the ratio

$$\frac{\theta^{b-a-1} P_\theta(X = a)}{P_\theta(X = b)}$$

is decreasing in θ.

This definition extends the notion of parametric monotone decreasing ratio introduced in Samaniego (1976) to a stronger property which is relevant in the estimation problem at hand.

Suppose a random sample $X_1, \ldots, X_n$ is obtained from a convoluted geometric distribution. Denote the likelihood function by

$$L(x_1, \ldots, x_n, \theta) = \prod_{i=1}^{n} P_\theta(X_i = x_i).$$

The characterization result implies that the likelihood equation $\frac{\partial}{\partial\theta} \ln L = 0$ may be written as

$$-\frac{n}{1-\theta} + \sum_{i=1}^{n} \sum_{j=1}^{x_i} \frac{\theta^{j-1} P_\theta(X_i = x_i - j)}{P_\theta(X_i = x_i)} = 0 \qquad (5)$$

It is clear from (5) that if a convoluted geometric distribution has strong PMDR, the derivative of the likelihood function is strictly decreasing and the likelihood equation has at most one solution. Thus, the maximum likelihood estimate of θ is either zero or the unique solution of the likelihood equation. The MLE may be approximated numerically to any desired degree of accuracy using the fact that the function $\frac{\partial}{\partial\theta} \ln L$ may cross zero only once.

The geometric distribution itself has strong PMDR. An example of a convoluted geometric distribution with strong PMDR is given below.

Example 1. Let $X = Y + Z$ where Y is geometric with parameter θ and Z is Bernoulli with known parameter p. Let $0 < a < b$, and consider

$$\frac{\theta^{b-a-1} P_\theta(X = a)}{P_\theta(X = b)} = \frac{\theta^{b-a-1}[(1-\theta)\theta^{a-1} p + (1-\theta)\theta^{a}(1-p)]}{(1-\theta)\theta^{b-1} p + (1-\theta)\theta^{b}(1-p)}$$

$$= \frac{(\frac{1}{\theta})\ p + \ (1-p)}{p + \theta\ (1-p)}$$

which is decreasing in θ. If $a = 0 < b$,

$$\frac{\theta^{b-a-1} P_\theta(X = a)}{P_\theta(X = b)} = \frac{1-p}{p + \theta(1-p)}$$

which is also decreasing in θ. Thus, this convoluted geometric distribution has the strong PMDR property.

There are convoluted geometric distributions for which the maximum likelihood estimation of θ poses an extremely cumbersome analytical problem. Examples can be constructed in which the likelihood equation has any fixed number of solutions. The example below illustrates maximum likelihood estimation of θ for a convoluted geometric distribution without strong PMDR.

Example 2. Let Z be a random variable with probability mass function

$$P(Z = n) = \begin{cases} \frac{1000}{2016} & \text{if } n = 0 \\ \frac{200}{2016} & \text{if } n = 1 \\ \frac{420}{2016} & \text{if } n = 2 \\ \frac{396}{2016} & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

Let Y be a geometric variable with parameter θ, and let $X = Y + Z$. Suppose one observation is taken on X, and $X = 3$ is observed. The equation $\frac{\partial}{\partial\theta} L = 0$ has solutions at $\theta = .1$, $.2$ and $.3$, the first and last corresponding to local maxima. The likelihood is maximized at both of these values, and either may serve as the MLE.

Since maximum likelihood estimation of θ is in general equivalent to finding zeros of a polynomial of degree $\sum_{i=1}^{n} x_i$, where $\underset{\sim}{x}$ is the vector of observations, the characterization result presented here, together with the strong PMDR property when it is applicable, provide a significant simplification of the problem.

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Let X be a nonnegative integer-valued random variable whose distribution is that of the sum of a geometric variable Y with parameter θ and a nonnegative integer-valued random variable Z independent of Y whose distribution does not depend on θ. The class of all such distributions are characterized by a system of differential equations satisfied by their probability mass functions. The characterization is shown to be useful in maximum likelihood estimation of the parameter θ when the distribution of Z is known.

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